low frequencies. Considerable departure from the quasi-static results has been shown to occur with increasing frequency, however. The analysis verifies the rise of characteristic impedance with frequency as predicted by Krage and Haddad [6].

REFERENCES

[1] T. Itoh and R. Mittra, "Dispersion characteristics of slot lines," *Electron. Lett.*, vol. 7, pp. 364–365, July 1971.
[2] ——, "Spectral-domain approach for calculating the dispersion characteristics of microstrip lines," *IEEE Trans. Microwave Theory Tech.* (Short Papers), vol. MTT-21, pp. 496–499, July 1973.
[3] ——, "A technique for computing dispersion characteristics of shielded microstrip lines," *IEEE Trans. Microwave Theory Tech.* (Short Papers), vol. MTT-22, pp. 896–898, Oct. 1974.
[4] T. Itoh, "Analysis of microstrip resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-22, pp. 946–952, Nov. 1974.
[5] J. B. Knorr and K.-D. Kuchler, "Analysis of coupled slots and coplanar strips on dielectric substrate," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 541–548, July 1975.
[6] M. K. Krage and G. I. Haddad, "Frequency-dependent characteristics of microstrip transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 678–688, Oct. 1972.
[7] H. A. Wheeler, "Transmission-line properties of parallel strips separated by a dielectric sheet," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 172–185, Mar. 1965.
[8] T. G. Bryant and J. A. Weiss, "Parameters of microstrip transmission lines and of coupled pairs of microstrip lines," *IEEE Trans. Microwave Theory Tech.* (*1968 Symposium Issue*), vol. MTT-16, pp. 1021–1027, Dec. 1968.
[9] W. J. Getsinger, "Microstrip dispersion model," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-21, pp. 34–39, Jan. 1973.
[10] J. W. Gould and E. C. Tolboys, "Even- and odd-mode guide wavelengths of coupled lines in microstrip," *Electron. Lett.*, vol. 8, pp. 121–122, Mar. 1972.
[11] W. J. Getsinger, "Dispersion of parallel-coupled microstrip," *IEEE Trans. Microwave Theory Tech.* (Short Papers), vol. MTT-21, pp. 144–145, Mar. 1973.

Circular Waveguide with Sinusoidally Perturbed Walls

OMAR RAFIK ASFAR AND ALI HASAN NAYFEH

Abstract—**Uniform second-order asymptotic expansions are obtained for the propagation of TM waves in a perfectly conducting circular waveguide with sinusoidally perturbed walls using the method of multiple scales. The analysis concerns the interaction of two propagating modes satisfying the resonance condition imposed by the periodicity of the waveguide walls. Two cases of resonance are treated as well as the case of decoupled modes. In the first case resonance occurs whenever the difference between the wavenumbers of the two interacting modes is nearly equal to the wall wavenumber, while in the second case the difference is nearly equal to twice the wall wavenumber. The results of the theory are then applied to the design of a mode coupler.**

I. INTRODUCTION

WAVEGUIDES having periodic structure find application in such microwave devices as the magnetron, the traveling-wave amplifier, and the linear accelerator [1]. In this paper we consider the case where the periodicity is a small perturbation of the waveguide wall which results in its use as a mode coupler.

We consider the case of propagation of TM modes in a perfectly conducting circular waveguide whose wall is sinusoidally perturbed so that the radius of the cross section of the guide at an axial location z' in a cylindrical coordinate system (ρ',ϕ,z') is given by

$$R(z') = R_0(1 + \epsilon \sin k_w'z') \quad (1)$$

Manuscript received December 6, 1974; revised March 27, 1975.
The authors are with The Department of Engineering Sciences and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, Va. 24061.

where R_0 is the average or unperturbed radius of the guide, k_w' is the wavenumber of the wall perturbation, and ϵ is a dimensionless parameter much smaller than unity and equal to the ratio of the amplitude of the periodic perturbation to the average radius R_0.

Marcuse and Derosier [2] treated the problem of a round dielectric waveguide with periodic wall corrugations and found that two guided modes are coupled if the difference between their wavenumbers is equal to the wavenumber of the wall k_w'. In fact, other resonances are possible as our analysis will show. They also confirmed the coupling experimentally and observed complete power conversion between the two modes. Marcuse used a combination of the Galerkin procedure and the method of averaging in order to obtain equations for the amplitudes of the interacting modes [3]. Chandezon *et al.* [4] treated wave propagation in a perfectly conducting guide with sinusoidally perturbed walls using the Rayleigh–Schrodinger technique to find a perturbation expansion in powers of ϵ for the case of a cylindrically symmetric TM mode $(\partial/\partial\phi = 0)$. They only considered the very special case of resonance when the wavenumber of the excited mode is equal to $\frac{1}{2}k_w'$ with a correction to first order in ϵ only.

A first-order uniformly valid expansion for the case of a parallel-plate waveguide with sinusoidal walls was obtained by Nayfeh and Asfar [5] for two interacting modes in the neighborhood of resonance that is given by the condition

$$k_w' \approx k_n' - k_s' \quad (2)$$

where $k_s'(k_n')$ are the wavenumbers of the generated (excited) mode. The resonances occur in the straightforward expansion as nonuniformities, and the uniformly valid expansion was obtained by using the method of multiple scales [6]. It was also shown that the solution is consistent with conservation of energy.

In this paper we determine second-order uniform expansions for two interacting modes satisfying the resonance condition (2) by using the method of multiple scales, as well as near the resonances

$$2k_w' \approx k_n' - k_s'. \tag{3}$$

The latter appears as a nonuniformity in the straightforward second-order equations which were found to be nonsolvable. Solutions away from resonance will also be obtained by using the method of multiple scales.

II. FORMULATION

We introduce dimensionless coordinates ρ and z given by $\rho = \rho'/R_0$, $z = z'/R_0$, and make time dimensionless by using R_0/c where c is the phase speed of electromagnetic waves in the medium. We consider the case of TM modes as an illustration of the technique which is applicable to TE modes as well. For TM modes the fields are derived from a z-directed vector potential whose scalar component ψ, which is assumed to have the time variation $\exp(-i\omega t)$, is governed by the Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0 \tag{4}$$

where k is the dimensionless free-space wavenumber and

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}. \tag{5}$$

The boundary condition is the vanishing of the tangential component of the electric field at $\rho = 1 + \epsilon \sin k_w z$. This gives

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi = -\epsilon k_w \cos k_w z \frac{\partial^2\psi}{\partial\rho\partial z}. \tag{6}$$

III. PERTURBATION EXPANSION USING THE METHOD OF MULTIPLE SCALES

We seek an asymptotic expansion to ψ in the form

$$\psi = \psi_0(\rho,\phi,Z_0,Z_1,Z_2) + \epsilon\psi_1(\rho,\phi,Z_0,Z_1,Z_2) + \epsilon^2\psi_2(\rho,\phi,Z_0,Z_1,Z_2) + \cdots \tag{7}$$

where $Z_0 = z$ is a short-length scale of the order of a wavelength; $Z_1 = \epsilon z$ and $Z_2 = \epsilon^2 z$ are long-length scales characterizing the spatial amplitude and phase modulations. Using the chain rule we can express $\partial/\partial z$ as

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial Z_0} + \epsilon\frac{\partial}{\partial Z_1} + \epsilon^2\frac{\partial}{\partial Z_2} + \cdots. \tag{8}$$

Substituting (7) and (8) into (4)–(6), expanding ψ at the boundary in a Taylor series around $\rho = 1$, and equating coefficients of equal powers of ϵ to zero, we obtain:

Order ϵ^0

$$\nabla_0^2\psi_0 + k^2\psi_0 = 0 \tag{9a}$$

where

$$\nabla_0^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial Z_0^2} \tag{9b}$$

$$\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\psi_0 = 0 \quad \text{at} \quad \rho = 1 \tag{9c}$$

Order ϵ

$$\nabla_0^2\psi_1 + k^2\psi_1 = -2\frac{\partial^2\psi_0}{\partial Z_0\partial Z_1} \tag{10a}$$

$$\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\psi_1 = -k_w \cos k_w Z_0 \frac{\partial^2\psi_0}{\partial\rho\partial Z_0} - \sin k_w Z_0\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\frac{\partial\psi_0}{\partial\rho} - 2\frac{\partial^2\psi_0}{\partial Z_0\partial Z_1} \quad \text{at} \quad \rho = 1 \tag{10b}$$

Order ϵ^2

$$\nabla_0^2\psi_2 + k^2\psi_2 = -2\frac{\partial^2\psi_0}{\partial Z_0\partial Z_2} - \frac{\partial^2\psi_0}{\partial Z_1^2} - 2\frac{\partial^2\psi_1}{\partial Z_0\partial Z_1} \tag{11a}$$

$$\begin{aligned}\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\psi_2 &= -\tfrac{1}{2}k_w \sin 2k_w Z_0 \frac{\partial^3\psi_0}{\partial\rho^2\partial Z_0} - k_w \cos k_w Z_0 \frac{\partial^2\psi_0}{\partial\rho\partial Z_1} \\ &\quad - 2\sin k_w Z_0 \frac{\partial^3\psi_0}{\partial\rho\partial Z_0\partial Z_1} - \sin^2 k_w Z_0 \cdot\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\frac{\partial^2\psi_0}{\partial\rho^2} \\ &\quad - \frac{\partial^2\psi_0}{\partial Z_1^2} - k_w \cos k_w Z_0 \frac{\partial^2\psi_1}{\partial\rho\partial Z_0} - 2\frac{\partial^2\psi_1}{\partial Z_0\partial Z_1} - 2\frac{\partial^2\psi_0}{\partial Z_0\partial Z_2} \\ &\quad - \sin k_w Z_0\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\frac{\partial\psi_1}{\partial\rho} \quad \text{at} \quad \rho = 1. \end{aligned} \tag{11b}$$

We assume that the solution of (9a) that is bounded at the axis is a linear combination of two propagating modes; i.e.,

$$\psi_0 = A_n(Z_1,Z_2)J_m(\gamma_{mn}\rho)\exp[i(k_n Z_0 + m\phi)] + A_s(Z_1,Z_2)J_m(\gamma_{ms}\rho)\exp[i(k_s Z_0 + m\phi)] \tag{12}$$

where J_m is Bessel's function of the first kind of order m and

$$k^2 - k_j^2 = \gamma_{mj}^2, \qquad j = n \quad \text{and} \quad s. \tag{13}$$

Substituting (12) into (9c), we have

$$J_m(\gamma_{mj}) = 0. \quad (14)$$

The functions A_n and A_s will be determined from the solvability conditions of the first- and second-order problems.

IV. SOLUTION OF FIRST-ORDER PROBLEM

Substituting (12) into (10a) and (10b) and using the fact that ψ_0 vanishes at the wall, we obtain

$$\nabla_0^2\psi_1 + k^2\psi_1 = -2i \sum_{j=n,s} k_j \frac{\partial A_j}{\partial Z_1} J_m(\gamma_{mj}\rho) \cdot \exp\left[i(k_jZ_0 + m\phi)\right] \quad (15a)$$

$$\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\psi_1 = \tfrac{1}{2}i \sum_{j=n,s} \gamma_{mj}A_jJ_m'(\gamma_{mj}) \cdot \{(\gamma_{mj}^2 - k_jk_w)\exp\left[i(k_j + k_w)Z_0\right] - (\gamma_{mj}^2 + k_jk_w)\exp\left[i(k_j - k_w)Z_0\right]\} \cdot \exp(im\phi) \quad \text{at} \quad \rho = 1. \quad (15b)$$

To determine the solution of (15a) and (15b), we distinguish between two cases: $k_n - k_s$ is away from k_w (the two modes are decoupled) and $k_n - k_s \approx k_w$ (resonance, the two modes are coupled). These cases are considered separately starting with the first.

Case of Decoupled Modes

In this case, the terms proportional to $\exp(ik_jZ_0)$, $\exp[i(k_j + k_w)Z_0]$, and $\exp[i(k_j - k_w)Z_0]$ are independent. Hence, we seek a particular solution to (15) in the form

$$\psi_1 = \Gamma_j(\rho)\exp\left[i(k_jZ_0 + m\phi)\right] + \tfrac{1}{2}i \sum_{j=n,s} \gamma_{mj}A_jJ_m'(\gamma_{mj}) \cdot \left[(\gamma_{mj}^2 - k_jk_w)\exp\left[i(k_j + k_w)Z_0\right]\Phi_{j1}(\rho) - (\gamma_{mj}^2 + k_jk_w)\exp\left[i(k_j - k_w)Z_0\right]\Theta_{j1}(\rho)\right]\exp(im\phi). \quad (16)$$

Substituting (16) into (15) and equating the coefficients of the independent terms in Z_0 on both sides; we obtain

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\Gamma_j}{d\rho}\right) + \left(\gamma_{mj}^2 - \frac{m^2}{\rho^2}\right)\Gamma_j = -2ik_j\frac{\partial A_j}{\partial Z_1}J_m(\gamma_{mj}\rho) \quad (17a)$$

$$\Gamma_j(\rho) = 0 \quad (17b)$$

$$\mathcal{L}(\Phi_{j1}) + \alpha_j^2\Phi_{j1} = 0 \quad \Phi_{j1}(1) = \alpha_j^{-2} \quad (18a)$$

where

$$\mathcal{L} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{m^2}{\rho^2} \quad \alpha_j^2 = k^2 - (k_j + k_w)^2 \quad (18b)$$

$$\mathcal{L}(\Theta_{j1}) + \beta_j^2\Theta_{j1} = 0 \quad \theta_{j1}(1) = \beta_j^{-2} \quad (19a)$$

where

$$\beta_j^2 = k^2 - (k_j - k_w)^2. \quad (19b)$$

Since the homogeneous parts of (17) have a nontrivial solution, the inhomogeneous problem has a solution if, and only if, a solvability condition is satisfied (i.e., the inhomogeneous parts are orthogonal to every solution of the adjoint homogeneous problem). Since (17a) and (17b) have the same form as (61) and (62), it follows from (66) that the solvability condition is

$$\frac{\partial A_j}{\partial Z_1} = 0 \quad \text{or} \quad A_j = A_j(Z_2) \quad (20)$$

hence, $\Gamma_j \equiv 0$.

The solutions of (18) and (19) are

$$\Phi_{j1}(\rho) = \left[\alpha_j^2J_m(\alpha_j)\right]^{-1}J_m(\alpha_j\rho) \quad (21)$$

$$\Theta_{j1}(\rho) = \left[\beta_j^2J_m(\beta_j)\right]^{-1}J_m(\beta_j\rho). \quad (22)$$

Substituting (21) and (22) into (17), and recalling that $\Gamma_j = 0$, we obtain

$$\psi_1 = i \sum_{j=n,s} A_j\{\Lambda_{1j}J_m(\alpha_j\rho)\exp\left[i(k_j + k_w)Z_0\right] + \Lambda_{2j}J_m(\beta_j\rho)\exp\left[i(k_j - k_w)Z_0\right]\}\exp(im\phi) \quad (23a)$$

where

$$\Lambda_{1j} = \tfrac{1}{2}\gamma_{mj}J_m'(\gamma_{mj})(\gamma_{mj}^2 - k_jk_w)\left[\alpha_j^2J_m(\alpha_j)\right]^{-1} \quad (23b)$$

$$\Lambda_{2j} = -\tfrac{1}{2}\gamma_{mj}J_m'(\gamma_{mj})(\gamma_{mj}^2 + k_jk_w)\left[\beta_j^2J_m(\beta_j)\right]^{-1}. \quad (23c)$$

Case of Coupled Modes

For the case of near resonance we let

$$k_s = k_n - k_w + \epsilon\delta \quad (24)$$

where $\delta = 0(1)$ is a detuning parameter, and express $\exp[i(k_n - k_w)Z_0]$ and $\exp[i(k_s + k_w)Z_0]$ as

$$\exp\left[i(k_n - k_w)Z_0\right] = \exp\left[i(k_sZ_0 - \delta Z_1)\right] \quad (25a)$$

$$\exp\left[i(k_s + k_w)Z_0\right] = \exp\left[i(k_nZ_0 + \delta Z_1)\right]. \quad (25b)$$

In this case, the terms proportional to $\exp(ik_sZ_0)$ and $\exp(ik_nZ_0)$ are not independent from the terms $\exp[i(k_n - k_w)Z_0]$ and $\exp[i(k_s + k_w)Z_0]$, respectively. Consequently, the solvability condition obtained in the preceding section has to be modified accordingly. To determine the solvability condition, we let

$$\psi_1 = \sum_{j=n,s} \Gamma_{j1}(\rho)\exp\left[i(k_jZ_0 + m\phi)\right]. \quad (26)$$

Substituting (26) into (15), using (25) and equating the coefficients of $\exp(ik_jZ_0)$ on both sides, we obtain

$$\mathcal{L}(\Gamma_{j1}) + \gamma_{mj}^2\Gamma_{j1} = -2ik_j\frac{\partial A_j}{\partial Z_1}J_m(\gamma_{mj}\rho) \quad (27)$$

$$\gamma_{mn}^2\Gamma_{n1} = \tfrac{1}{2}i\gamma_{ms}A_sJ_m'(\gamma_{ms})(\gamma_{ms}^2 - k_sk_w)\exp(i\delta Z_1) \quad \text{at} \quad \rho = 1 \quad (28a)$$

$$\gamma_{ms}^2\Gamma_{s1} = -\tfrac{1}{2}i\gamma_{mn}A_nJ_m'(\gamma_{mn})(\gamma_{mn}^2 + k_nk_w)\exp(-i\delta Z_1) \quad \text{at} \quad \rho = 1. \quad (28b)$$

Since (27) and (28) have the same form as (61) and (62), it follows from (66) that their solvability conditions are

$$\frac{\partial A_n}{\partial Z_1} = P_1 \exp(i\delta Z_1) A_s \quad (29a)$$

$$\frac{\partial A_s}{\partial Z_1} = P_2 \exp(-i\delta Z_1) A_n \quad (29b)$$

where

$$P_1 = \tfrac{1}{2}(\gamma_{ms}/k_n\gamma_{mn}) J_m'(\gamma_{ms})(\gamma_{ms}^2 - k_s k_w)/J_m'(\gamma_{mn}) \quad (29c)$$

$$P_2 = -\tfrac{1}{2}(\gamma_{mn}/k_s\gamma_{ms}) J_m'(\gamma_{mn})(\gamma_{mn}^2 + k_n k_w)/J_m'(\gamma_{ms}). \quad (29d)$$

Equations (29a) and (29b) admit solutions of the form

$$A_n = a_n \exp[i(q+\delta)Z_1] \quad A_s = a_s \exp(iqZ_1) \quad (30a)$$

where a_n and a_s are constants and

$$q = \tfrac{1}{2}[-\delta \pm (\delta^2 - 4P_1P_2)^{1/2}]. \quad (30b)$$

Since $\gamma_{ms}^2 - k_s k_w = k^2 - k_s^2 - k_s k_w = k^2 - k_s(k_s + k_w) = k^2 - k_s(k_n + \epsilon\delta)$, and since $k > k_s$ and $k_n, P_1P_2 < 0$, hence q is real and A_n and A_s are oscillatory. Therefore, to this order, (30a) shows that neither mode can exist in the waveguide without strongly exciting the other and that both modes propagate unattenuated along the waveguide with a continuous energy exchange between the interacting modes.

The inhomogeneous solution for ψ_1 in this case can be abstracted from (23) by excluding the terms $\exp[i(k_n - k_w)Z_0]$ and $\exp[i(k_s + k_w)Z_0]$; that is

$$\psi_1 = i\Lambda_{1n} A_n J_m(\alpha_n \rho) \exp[i(k_n + k_w)Z_0 + im\phi] + i\Lambda_{2s} A_s J_m(\beta_s \rho) \exp[i(k_s - k_w)Z_0 + im\phi]. \quad (31)$$

V. SOLUTION OF SECOND-ORDER PROBLEM

To determine the variation of A_j with Z_2, we need to investigate the second-order problem. To accomplish this, we distinguish three cases:

1) the case of decoupled modes (i.e., $k_n - k_s$ away from k_w or $2k_w$);
2) the case of second-order resonance ($k_n - k_s \approx 2k_w$);
3) the case of first-order resonance treated in the previous section ($k_n - k_s \approx k_w$).

In what follows, we treat each of these cases separately.

Case of Decoupled Modes

Substituting (12) and (23) into (11a) and (11b), and using the fact that $\partial A_j/\partial Z_1 = 0$, we obtain

$$\nabla_0^2\psi_2 + k^2\psi_2 = -2i \sum_{j=n,s} k_j \frac{dA_j}{dZ_2} J_m(\gamma_{mj}\rho) \cdot \exp[i(k_j Z_0 + m\phi)] \quad (32a)$$

$$\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\psi_2 = \sum_{j=n,s} \{E_{0j}\exp(ik_jZ_0) + E_{1j}\exp[i(k_j + 2k_w)Z_0] + E_{2j}\exp[i(k_j - 2k_w)Z_0]\} A_j \exp(im\phi) \quad \text{at} \quad \rho = 1 \quad (32b)$$

where the constants E_{ij} are defined in Appendix II.

As before, we determine the solvability condition for (32a) and (32b) by seeking a particular solution of the form

$$\psi_{2p} = \sum_{j=n,s} \Gamma_{j2}(\rho) \exp[i(k_j Z_0 + m\phi)]. \quad (33)$$

Substituting (33) into (32a) and (32b), and equating the coefficients of $\exp(ik_jZ_0)$ on both sides, we obtain

$$\mathcal{L}(\Gamma_{j2}) + \gamma_{mj}^2 \Gamma_{j2} = -2ik_j \frac{dA_j}{dZ_2} J_m(\gamma_{mj}\rho) \quad (34a)$$

$$\Gamma_{j2} = \gamma_{mj}^{-2} E_{0j} A_j \quad \text{at} \quad \rho = 1. \quad (34b)$$

Since (34a) and (34b) have the same form as (61) and (62), it follows from (66) that the solvability condition is

$$\frac{dA_j}{dZ_2} = ik_j^{-1}\gamma_{mj}^{-1} E_{0j} A_j / J_m'(\gamma_{mj}). \quad (35)$$

The solution for A_j is therefore

$$A_j = a_j \exp[-i(E_{0j}/k_j J_m'(\gamma_{mj})\gamma_{mj}) Z_2] \quad (36)$$

where a_j is a constant. We note that the exponent in (36) is pure imaginary and hence A_j is bounded. Thus the effect of the wall perturbations is to decrease the wavenumbers of all decoupled modes.

Case of $k_n - k_s \approx 2k_w$

Substituting (12) and (23) into (11a) and (11b), and using the fact that $\partial A_j/\partial Z_1 = 0$, we obtain

$$\nabla_0^2\psi_2 + k^2\psi_2 = -2i \sum_{j=n,s} k_j \frac{dA_j}{dZ_2} J_m(\gamma_{mj}\rho) \cdot \exp[i(k_j Z_0 + m\phi)] \quad (37)$$

$$\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\psi_2 = \sum_{j=n,s} \{E_{0j}\exp(ik_jZ_0) + E_{1j}\exp[i(k_j + 2k_w)Z_0] + E_{2j}\exp[i(k_j - 2k_w)Z_0]\} A_j \exp(im\phi) \quad \text{at} \quad \rho = 1 \quad (38)$$

where the constants E_{ij} are given in Appendix II.

For the case of near resonance, we let

$$k_n = k_s + 2k_w + \epsilon^2\delta \quad (39)$$

where δ is a detuning parameter. Using (39), we express $\exp[i(k_n - 2k_w)Z_0]$ and $\exp[i(k_s + 2k_w)Z_0]$ as

$$\exp[i(k_n - 2k_w)Z_0] = \exp[i(k_s Z_0 + \delta Z_2)]$$
$$\exp[i(k_s + 2k_w)Z_0] = \exp[i(k_n Z_0 - \delta Z_2)].$$

Substituting these into (38), we obtain

$$\left(\frac{\partial^2}{\partial Z_0^2} + k^2\right)\psi_2 = \{[\sum_{j=n,s} A_j E_{0j} \exp(ik_j Z_0) + A_s E_{1s} \exp(-i\delta Z_2) \cdot \exp(ik_n Z_0) + A_n E_{2n} \exp(i\delta Z_2) \exp(ik_s Z_0)] + A_n E_{1n} \exp[i(k_n + 2k_w)Z_0] + A_s E_{2s} \cdot \exp[i(k_s - 2k_w)Z_0]\} \exp(im\phi) \quad \text{at} \quad \rho = 1. \tag{40}$$

As outlined in the previous sections, we can find the following solvability condition for (37) and (40):

$$\frac{dA_n}{dZ_2} = -i[k_n \gamma_{mn} J_m'^2(\gamma_{mn})]^{-1}[E_{0n}A_n + E_{1s} \cdot \exp(-i\delta Z_2)A_s] \tag{41}$$

$$\frac{dA_s}{dZ_2} = -i[k_s \gamma_{ms} J_m'^2(\gamma_{ms})]^{-1}[E_{0s}A_s + E_{2n} \cdot \exp(i\delta Z_2)A_n]. \tag{42}$$

We seek the solution of (41) and (42) in the form

$$A_s = a_s \exp(qZ_2) \quad A_n = a_n \exp[(q - i\delta)Z_2]. \tag{43}$$

Substituting (43) into (41) and (42) and eliminating the a's, we obtain

$$q^2 + i\sigma q + \Omega = 0 \tag{44}$$

where

$$\sigma = E_{0s}[k_s \gamma_{ms} J_m'^2(\gamma_{ms})]^{-1} - \delta + E_{0n}[k_n \gamma_{mn} J_m'^2(\gamma_{mn})]^{-1} \tag{45a}$$

and

$$\Omega = \delta E_{0s}[k_s \gamma_{ms} J_m'^2(\gamma_{ms})]^{-1} + [k_n k_s \gamma_{mn} \gamma_{ms} J_m'^2(\gamma_{mn}) J_m'^2 \cdot (\gamma_{ms})]^{-1}[E_{1s}E_{2n} - E_{0n}E_{0s}]. \tag{45b}$$

The solution of (44) is

$$q = -\tfrac{1}{2}i[\sigma^2 \pm (\sigma^2 + 4\Omega)^{1/2}]. \tag{46}$$

Using the expressions for the E_{ij} from Appendix II, one can show that $\sigma^2 + 4\Omega > 0$ so that q is pure imaginary. Consequently, A_n and A_s are oscillatory and neither mode can exist in the waveguide without exciting the other. Moreover, both modes propagate unattenuated along the waveguide with the energy being continuously exchanged between them.

Case of $k_n - k_s \approx k_w$

Substituting (12) and (31) into (11a) and (11b), and using (24), we obtain

$$\nabla_0^2\psi_2 + k^2\psi_2 = \left\{-\sum_{j=n,s}\left[2ik_j \frac{\partial A_j}{\partial Z_2} + \frac{\partial^2 A_j}{\partial Z_1^2}\right] J_m(\gamma_{mj}\rho) \exp(ik_j Z_0) + B_n J_m(\alpha_n \rho) A_s \exp[i(k_s + 2k_w)Z_0] + B_s J_m(\beta_s \rho) A_n \exp[i(k_n - 2k_w)Z_0]\right\} \exp(im\phi) \tag{47}$$

$$\left[\frac{\partial^2}{\partial Z_0^2} + k^2\right]\psi_2 = \{D_{0n}A_n \exp(ik_n Z_0) + D_{0s}A_s \exp(ik_s Z_0) + D_{1n}A_n \exp[i(k_n + 2k_w)Z_0] + D_{1s}A_s \exp[i(k_s + 2k_w)Z_0] + D_{2n}A_n \exp[i(k_n + 2k_w)Z_0] + D_{2s}A_s \exp[i(k_s - 2k_w)Z_0]\} \exp(im\phi) \quad \text{at} \quad \rho = 1 \tag{48}$$

where the constants D_{0s} and D_{0n} are defined in Appendix II, and the constants B_j, and D_{1j}, and D_{2j} do not affect $\partial A_j/\partial Z_2$ as evident from the solvability condition which will be given.

We find the solvability condition for (47) and (48) by seeking a particular solution of the form (33) and following the procedure outlined in the previous section; the result is

$$\frac{\partial^2 A_j}{\partial Z_1^2} + 2ik_j \frac{\partial A_j}{\partial Z_2} = 2P_{0j}A_j \tag{49}$$

where

$$P_{0j} = D_{0j}/\gamma_{mj} J_m'(\gamma_{mj}). \tag{50}$$

In order to find A_j we have to solve the system (29) and (49). Thus differentiating (29a) and (29b) with respect to Z_1 and substituting into (49), we obtain

$$\frac{\partial A_n}{\partial Z_2} = -\tfrac{1}{2}(\delta/k_n)P_1 \exp(i\delta Z_1)A_s - i[(P_{0n} - \tfrac{1}{2}P_1P_2)/k_n]A_n \tag{51}$$

$$\frac{\partial A_s}{\partial Z_2} = \tfrac{1}{2}(\delta/k_s)P_2 \exp(-i\delta Z_1)A_n - i[(P_{0s} - \tfrac{1}{2}P_1P_2)/k_s]A_s. \tag{52}$$

We note here that (29a) and (51) are the first two terms in a multiple space scaling analysis of

$$\frac{dA_n}{dz} = \epsilon[1 - \tfrac{1}{2}\epsilon(\delta/k_n)]P_1 \exp(i\epsilon\delta z)A_s - i\epsilon^2[(P_{0n} - \tfrac{1}{2}P_1P_2)/k_n]A_n. \tag{53}$$

Similarly, (29b) and (52) are the first two terms in a multiple scaling analysis of

$$\frac{dA_s}{dz} = \epsilon[1 + \tfrac{1}{2}\epsilon(\delta/k_s)]P_2 \exp(-i\epsilon\delta z)A_n - i\epsilon^2[(P_{0s} - \tfrac{1}{2}P_1P_2)/k_s]A_s. \quad (54)$$

We now seek the solution of (53) and (54) in the form

$$A_n = a_n \exp[i(q + \epsilon\delta)z] \quad A_s = a_s \exp(iqz) \quad (55)$$

where a_n, a_s, and q are constants. Substituting (55) into (53) and (54), and eliminating the a's we obtain

$$q^2 + \xi q - \Omega = 0 \quad (56)$$

where

$$\xi = \delta\epsilon + \epsilon^2[k_n^{-1}(P_{0n} - \tfrac{1}{2}P_1P_2) + k_s^{-1}(P_{0s} - \tfrac{1}{2}P_1P_2)] \quad (57a)$$

$$\Omega = \epsilon^2 P_1P_2(k_nk_s)^{-1}[k_nk_s - \tfrac{1}{4}\epsilon^2\delta^2 + \tfrac{1}{2}\epsilon\delta(k_n - k_s)] - \epsilon^3(\delta/k_s)(P_{0s} - \tfrac{1}{2}P_1P_2) - \epsilon^4(k_nk_s)^{-1} \cdot (P_{0n} - \tfrac{1}{2}P_1P_2)(P_{0s} - \tfrac{1}{2}P_1P_2). \quad (57b)$$

The solution of (56) is

$$q = -\tfrac{1}{2}[\xi \pm (\xi^2 + 4\Omega)^{1/2}]. \quad (58)$$

Since P_1P_2 is negative as shown in Section IV, $\Omega = -\epsilon^2P_1P_2 + 0(\epsilon^3)$ according to (57b); Ω is positive. Hence, q is real and A_n and A_s are oscillatory as a consequence. Thus the interacting modes propagate unattenuated along the waveguide with the energy being continuously exchanged between them.

VI. DESIGN OF A MODE COUPLER

The foregoing analysis applies to any Fourier component of a wall distortion function that is a general periodic function. Assuming that the excitation at $Z = 0$ is given as $A_n(0) = 1$ and $A_s(0) = 0$, then using the results of first-order coupling of Section IV and following the procedure for the derivation of (57a) and (57b) of [5], we obtain

$$|A_n| = (\delta^2 - 4P_1P_2)^{-1/2}\{\delta^2 - 4P_1P_2 \cdot \cos^2[\tfrac{1}{2}(\delta^2 - 4P_1P_2)^{1/2}Z_1]\}^{1/2} \quad (59)$$

$$|A_s| = 2(\delta^2 - 4P_1P_2)^{-1/2}P_2 \cdot \sin[\tfrac{1}{2}(\delta^2 - 4P_1P_2)^{1/2}Z_1]. \quad (60)$$

If these expressions are now substituted into (12) and then the Poynting vector is evaluated and integrated over the cross section of the guide, one can show that energy is conserved to $0(\epsilon)$. In other words, the energy of the excitation at $Z = 0$ is equal to the sum of the energies of the two interacting modes at any location along the z axis. Equations (59) and (60) show that the energy will be exchanged between the two modes along the guide. This feature of energy exchange can be utilized in the design of mode couplers operating under the resonance condition $k_n - k_s \approx k_w$ as will be discussed.

We consider a waveguide whose TM_{01} mode is cut off at 5 GHz. At 15 GHz, the wavenumbers of the TM_{01} and TM_{02} modes are 6.8 and 4.64, respectively. Then, if the wall perturbation has a wavenumber approximately equal to 2.16, these two modes are strongly coupled. In what follows, we consider a wall perturbation with $\epsilon = 0.1$ and $k_w = 2.16$ corresponding to $\delta = 0$, and $k_w = 2.06$ corresponding to $\delta = 1.0$. Then the variation of the amplitudes of the two coupled modes are calculated from (59) and (60).

Figs. 1 and 2 show that complete energy exchange is achieved between the two modes only when the modes are perfectly tuned (i.e., $\delta \equiv 0$). When the TM_{01} mode is excited at $Z = 0$, Fig. 1 for $\delta = 0$ shows that all the energy is transferred to the TM_{02} mode at $Z_1 = 0.86$ corresponding to the length $8.6R_0$. When $\delta = 1.0$, Fig. 2 shows that the maximum energy transfer occurs at $Z_1 = 0.84$ corresponding to the length $8.4R_0$.

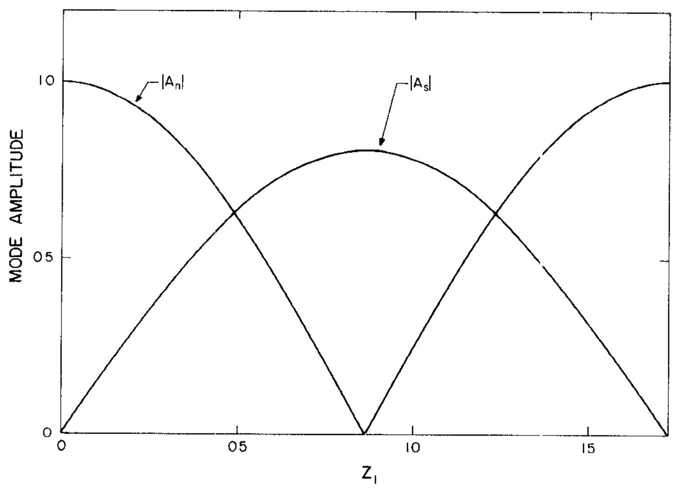

Fig. 1. Variation of mode amplitudes $|A_n|$ and $|A_s|$ along the axis for perfectly tuned modes.

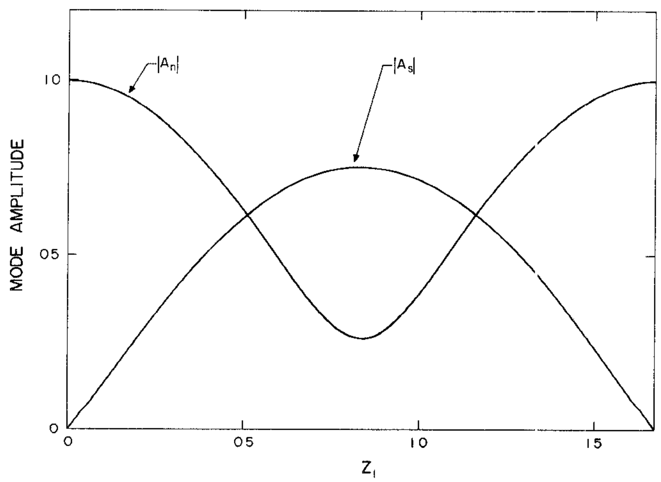

Fig. 2. Variation of mode amplitudes $|A_n|$ and $|A_s|$ along the axis for detuned modes with $\delta = 1.0$.

APPENDIX I

To determine the solvability condition for a problem of the form

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\Gamma_j}{d\rho}\right) + \left(\gamma_{mj}^2 - \frac{m^2}{\rho^2}\right)\Gamma_j = F_j(\rho) \qquad (61)$$

$$\Gamma_j(1) = c_j \qquad (62)$$

we multiply (61) by a function $\rho u(\rho)$, to be specified later, integrate the result by parts from $\rho = 0$ to $\rho = 1$, and obtain

$$\int_0^1 \rho\Gamma_j\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) + \left(\gamma_{mj}^2 - \frac{m^2}{\rho^2}\right)u\right]d\rho$$

$$+ u(1)\frac{d\Gamma_j}{d\rho}(1) - \Gamma_j(1)\frac{du}{d\rho}(1)$$

$$= \int_0^1 \rho u(\rho)F_j(\rho)\, d\rho. \qquad (63)$$

We choose $u(\rho)$ to be a solution of the so-called adjoint homogeneous problem

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) + \left(\gamma_{mj}^2 - \frac{m^2}{\rho^2}\right)u = 0 \qquad (64)$$

$$u(1) = 0. \qquad (65)$$

We take the solution of (64) and (65) that is bounded at $\rho = 0$ as $u(\rho) = J_m(\gamma_{mj}\rho)$. Substituting for u into (63) and using the boundary condition (62), we arrive at the following solvability condition:

$$c_j\gamma_{mj}J_m'(\gamma_{mj}) + \int_0^1 \rho J_m(\gamma_{mj}\rho)F_j(\rho)\, d\rho = 0. \qquad (66)$$

APPENDIX II

$$E_{0j} = -\tfrac{1}{2}\gamma_{mj}^4 J_m''(\gamma_{mj}) + \tfrac{1}{4}\gamma_{mj}J_m'(\gamma_{mj})$$
$$\cdot\{(\gamma_{mj}^2 - k_jk_w)^2[J_m'(\alpha_j)/\alpha_jJ_m(\alpha_j)]$$
$$+ (\gamma_{mj}^2 + k_jk_w)^2[J_m'(\beta_j)/\beta_jJ_m(\beta_j)]\}$$

$$E_{1j} = \tfrac{1}{4}(\gamma_{mj}^2 - k_jk_w)\{\gamma_{mj}^2J_m''(\gamma_{mj}) - (\gamma_{mj}/\alpha_j)$$
$$\cdot(\gamma_{mj}^2 - 2k_w^2 - 3k_jk_w)[J_m'(\gamma_{mj})J_m'(\alpha_j)/J_m(\alpha_j)]\}$$

$$E_{2j} = \tfrac{1}{4}(\gamma_{mj}^2 + k_jk_w)\{\gamma_{mj}^2J_m''(\gamma_{mj}) - (\gamma_{mj}/\beta_j)$$
$$\cdot(\gamma_{mj}^2 - 2k_w^2 + 3k_jk_w)[J_m'(\gamma_{mj})J_m'(\beta_j)/J_m(\beta_j)]\}$$

$$D_{0n} = \tfrac{1}{4}\gamma_{mn}\{J_m'(\gamma_{mn})([k_s(\gamma_{mn}^2 + k_nk_w)(2k_s + k_w)$$
$$+ (\gamma_{mn}^2 - k_nk_w)^2[J_m'(\alpha_n)/\alpha_nJ_m(\alpha_n)])$$
$$- 2\gamma_{mn}^3J_m''(\gamma_{mn})\}$$

$$D_{0s} = \tfrac{1}{4}\gamma_{ms}\{J_m'(\gamma_{ms})([k_n(\gamma_{ms}^2 - k_sk_w)(2k_n - k_w)$$
$$+ (\gamma_{ms}^2 + k_sk_w)^2[J_m'(\beta_s)/\beta_sJ_m(\beta_s)])$$
$$- 2\gamma_{ms}^3J_m''(\gamma_{ms})\}.$$

REFERENCES

[1] J. C. Slater, *Microwave Electronics.* New York: Dover, 1969, ch. 8.
[2] D. Marcuse and R. Derosier, "Mode conversion caused by diameter changes of a round dielectric waveguide," *Bell Syst. Tech. J.*, Dec. 1969.
[3] D. Marcuse, *Light Transmission Optics.* New York: Van Nostrand-Reinhold, 1972, ch. 9.
[4] J. Chandezon, G. Cornet, and G. Raoult, "Propagation des ondes dans les guides cylindrique à génératrice sinusoïdale. Expressions des champs," *C. R. Acad. Sci.*, Ser. B, 277, 1973, p. 355.
[5] A. H. Nayfeh and O. R. Asfar, "Parallel-plate waveguide with sinusoidally perturbed boundaries," *J. Appl. Phys.*, vol. 45, p. 4797, 1974.
[6] A. H. Nayfeh, *Perturbation Methods.* New York: Wiley-Interscience, 1973, ch. 6.

Asymmetric Coupled Transmission Lines in an Inhomogeneous Medium

VIJAI K. TRIPATHI, MEMBER, IEEE

Abstract—Terminal characteristic parameters for a uniform coupled-line four-port for the general case of an asymmetric, inhomogeneous system are derived in this paper. The parameters (impedance, admittance, etc.) are derived in terms of two independent modes that propagate in two uniformly coupled propagating systems. The four-port parameters derived are of the same form as those obtained for the symmetric case resulting in similar two-port equivalent circuits for various circuit configurations considered by Zysman and Johnson [1]. The results obtained should be quite useful in designing asymmetric coupled-line circuits in an inhomogeneous medium for various known applications.

Manuscript received November 13, 1974; revised April 15, 1975.
The author is with the Department of Electrical and Computer Engineering, Oregon State University, Corvallis, Oreg. 97331.

INTRODUCTION

UNIFORM coupled-line circuits are used for many applications including filters, couplers, and impedance matching networks. These circuits are usually